EXPANSION OF A CIRCULAR OPENING IN A RIGID-PLASTIC PLATE

(RASSHIRENIE KRUGOVOGO OTVERSTIIA V ZHESTKO-PLASTICHESKOI PLASTINKE)

PMM Vol.25, No.3, 1961, pp. 548-552

V. V. SOKOLOVSKII
(Moscow)

(Received November 27, 1960)

This paper treats a problem of expansion of a circular opening in an infinite rigid-plastic plate, taking into account the increase of its thickness near the contour. This problem was formulated and solved by Taylor and Hill [1] using the Tresca plasticity condition.

Let us analyse the problem of expansion of a circular opening using the plasticity condition

$$\sigma_r^2 - \sigma_r\sigma_\theta + \sigma_\theta^2 = 3k^2$$

and the associated flow law.

Three annular regions must be distinguished here: a plastic region $a \leqslant r \leqslant b$ with a varying thickness of the plate, h; a plastic region $b \leqslant r \leqslant c$ with a constant thickness h_0; and a rigid region $c \leqslant r \leqslant \infty$ with a constant thickness h_0, as shown in Fig. 1.

The equilibrium equation for a plate of thickness h is known to be

$$\frac{\partial (h\sigma_r)}{\partial r} + \frac{h(\sigma_r - \sigma_\theta)}{r} = 0$$

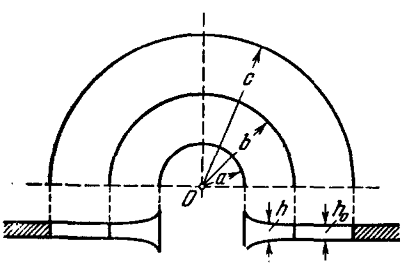

Fig. 1.

The usual relationship between the stress components and strain rates

$$\frac{\varepsilon_r}{2\sigma_r - \sigma_\theta} = \frac{\varepsilon_\theta}{2\sigma_\theta - \sigma_r}$$

immediately yields the following:

$$\frac{\partial v}{\partial r} = \frac{2\sigma_r - \sigma_\theta}{2\sigma_\theta - \sigma_r}\frac{v}{r}$$

The condition of incompressibility can be expressed as

$$\frac{1}{h}\left(\frac{\partial h}{\partial c}+v\frac{\partial h}{\partial r}\right)+\frac{\partial v}{\partial r}+\frac{v}{r}=0$$

where c determines the time scale. Let

$$\alpha = a / c, \qquad \beta = b / c, \qquad \rho = r / c$$

We shall assume that all unknown quantities are functions of ρ only.

Consider first the annular region $\alpha \leqslant \rho \leqslant \beta$ where h is variable, and let us investigate the basic equations describing the plastic flow in this region. The equilibrium equation is now written as

$$\frac{d(h\sigma_r)}{d\rho}+\frac{h(\sigma_r-\sigma_\theta)}{\rho}=0 \tag{1}$$

and the plasticity condition as

$$\sigma_r^2-\sigma_r\sigma_\theta+\sigma_\theta^2=3k^2 \tag{2}$$

The equations relating radial velocity with the stress components are

$$\frac{dv}{d\rho}=\frac{2\sigma_r-\sigma_\theta}{2\sigma_\theta-\sigma_r}\frac{v}{\rho} \tag{3}$$

and the condition of incompressibility

$$\frac{v-\rho}{h}\frac{dh}{d\rho}+\frac{dv}{d\rho}+\frac{v}{\rho}=0 \tag{4}$$

Let us express the stress components σ_r and σ_θ in terms of a new variable as follows:

$$\left.\begin{matrix}\sigma_r\\ \sigma_\theta\end{matrix}\right\}=-2k\sin\left(\varphi\pm\frac{\pi}{6}\right)$$

and let

$$w=\frac{v}{\rho}, \qquad \psi=\varphi+\frac{\pi}{6}$$

Equations (1) and (2), together with (3) and (4), result in

$$\rho\frac{d\varphi}{d\rho}=-\frac{\sqrt{3}\,w-2\cos\varphi\cos\psi}{2(w-1)\cos^2\psi}$$
$$\rho\frac{dw}{d\rho}=-\frac{\sqrt{3}\cos\varphi}{\cos\psi}w \tag{5}$$

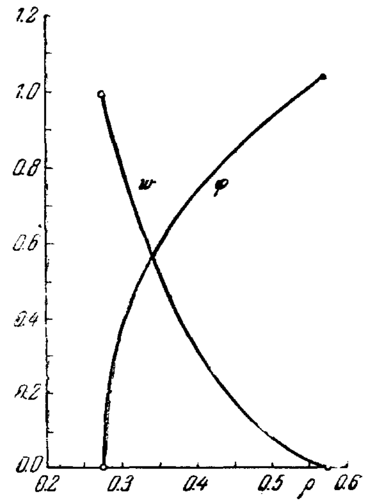

Fig. 2.

and in

$$\frac{\rho}{h}\frac{dh}{d\rho}=\frac{w\sin\varphi}{(w-1)\cos\psi} \tag{6}$$

Equation (5), after elimination of w, results in a differential equation of second order. Changing the variables

$$\rho\frac{d\varphi}{d\rho}=\frac{\tan^2\psi}{\omega}\left(\frac{\sin\psi}{\sin\varphi}\right)^2\exp(\sqrt{3}\,\varphi), \qquad \Phi=\frac{\sqrt{3}}{2}\left(\frac{\sin\varphi}{\sin\psi}\right)^3\exp(-\sqrt{3}\,\varphi)$$

this equation reduces to the Abel equation. We finally have

$$\frac{1}{\omega}\frac{d\omega}{d\varphi}=\frac{2\cos^2\varphi\cos^2\psi}{\sin^3\varphi\sin^3\psi}\Phi^2\omega^2+\frac{2(1+\cos^2\varphi)\cos^2\psi-\cos\varphi\sin\psi}{\sin^2\varphi\sin^2\psi}\Phi\omega$$

In the vicinity of points $\rho = \alpha$ and $\rho = \beta$ we can deduce approximate integrals of (5) and (6). In the neighborhood of a point $\rho = \alpha$, $\phi = 0$ it is easy to obtain

$$\varphi=C\xi^{1/3}, \qquad w=1-2\xi, \qquad \frac{h}{h_0}=D(1-\sqrt{3}\,C\xi^{1/3}) \tag{7}$$

and near the point $\rho = \beta$, $\phi = \pi/3$ it is easy to find

$$\varphi=\frac{\pi}{3}+\frac{\sqrt{3}}{2}\left(\eta+\frac{1}{4}\eta^2\right), \qquad w=-\frac{1}{2}\eta+\frac{11}{8}\eta^2, \qquad \frac{h}{h_0}=1-\frac{1}{2}\eta+\frac{5}{8}\eta^2 \tag{8}$$

where

$$\xi=\frac{\rho}{\alpha}-1, \qquad \eta=\frac{\rho}{\beta}-1$$

Differential equations (5) and (6) permit the construction of functions ϕ, w and h inside of an interval $\alpha \leqslant \rho \leqslant \beta$. The approximate integrals (7) and (8) permit us to obtain the same integrals near points $\rho = \alpha$ and $\rho = \beta$. The integral curves of ϕ and w are shown in Fig. 2 for $\rho = 0.276$, $C = 0.78$, $D = 3.61$ and $h = 3.61\ h_0$ at $\rho = \alpha$.

It has to be noted that near the contour of the opening, where h is varying quite rapidly, the theory proposed above cannot claim any high degree of accuracy.

Let us analyse now an annular region $\beta \leqslant \rho \leqslant 1$ where h is constant. Obviously, we have here the known solution satisfying the condition $\sigma_r + \sigma_\theta = 0$ or $\phi = 0$ for $\rho = 1$. This solution is

$$\rho=\frac{1}{\sqrt{\cos\varphi}}\exp\left(-\frac{\sqrt{3}}{2}\varphi\right), \qquad w=v=0, \qquad h=h_0$$

where $0 \leqslant \phi \leqslant \pi/3$.

The value of $\rho = \beta$ corresponding to $\rho = \pi/3$ is

$$\beta = \sqrt{2}\exp\left(-\frac{\pi}{2\sqrt{3}}\right) = 0.571$$

The results of an approximate integration of (5) and (6) are shown in Table 1. They are applicable to the whole annular region $a \leqslant \rho \leqslant \beta$ where the thickness of the plate is variable.

The dependence of the stresses σ_r and σ_θ on ρ are shown by solid lines in Fig. 3, and the dependence of h by a solid line in Fig. 4. Both σ_r and σ_θ are continuous in $a \leqslant \rho \leqslant 1$.

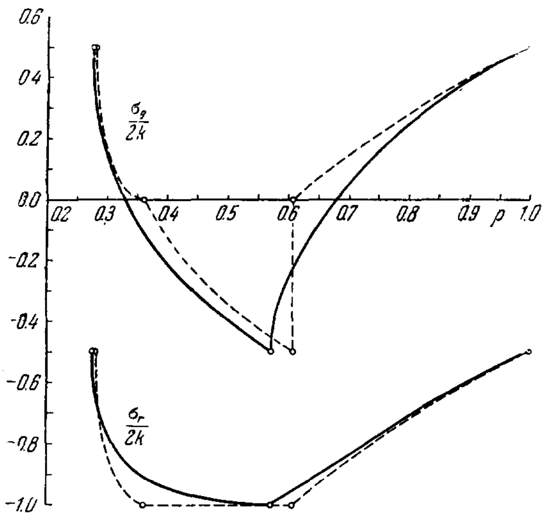

Fig. 3.

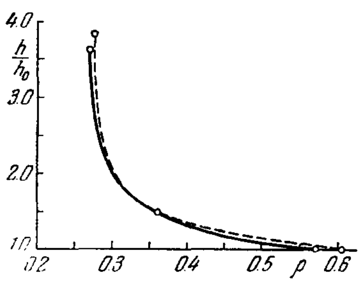

Fig. 4.

Consider, for the sake of comparison, the same problem using the plasticity condition

$$\sigma_\theta - \sigma_r = 2k, \qquad \sigma_r\sigma_\theta \leqslant 0, \qquad \sigma_r = -2k, \qquad \sigma_r\sigma_\theta \geqslant 0$$

and the usual relationships between the stress components and strain rate components, as was done by Hill.

First consider an annular region $a \leqslant \rho \leqslant \beta$, where the thickness of the plate is variable, and let us investigate basic relationships which describe plastic flow in a part of this region where $\sigma_r = -2k$. Equation (1) and plasticity condition $\sigma_r = -2k$, together with (3) and (4), establish

$$\rho\frac{dw}{d\rho} = \frac{3(1-\sigma)}{2\sigma-1}w, \qquad w = \frac{(2\sigma-1)(\sigma-1)}{2(\sigma^2-\sigma+1)}, \qquad \sigma = -\frac{\sigma_\theta}{2k} \tag{9}$$

and therefore

$$\frac{\rho}{h}\frac{dh}{d\rho} = \sigma - 1 \tag{10}$$

Eliminating w from (9) results in

$$\rho \frac{d}{d\rho}\left[\frac{(2\sigma-1)(\sigma-1)}{\sigma^2-\sigma+1}\right] = -\frac{3(\sigma-1)^2}{\sigma^2-\sigma+1}$$

which can be integrated in a closed form. Since $\sigma = 1/2$, for $\rho = \beta$, we have

$$\rho = \beta \frac{\sqrt{\sigma^2-\sigma+1}}{\sqrt{3}(1-\sigma)} \exp\left[\frac{1-2\sigma}{3(1-\sigma)} - \frac{1}{\sqrt{3}} \tan^{-1} \frac{1-2\sigma}{\sqrt{3}}\right] \tag{11}$$

and therefore

$$w = \frac{(2\sigma-1)(\sigma-1)}{2(\sigma^2-\sigma+1)}, \qquad h = \frac{h_0}{[2(1-\sigma)]^{1/3}} \exp\left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{1-2\sigma}{\sqrt{3}}\right] \tag{12}$$

TABLE 1.

ρ	φ	w	$-\sigma_r/2k$	$\sigma_\theta/2k$	v	h/h_0
0.276	0.000	1.000	0.500	0.500	0.276	3.61
0.280	0.188	0.970	0.653	0.329	0.272	2.67
0.290	0.309	0.871	0.739	0.213	0.253	2.26
0.300	0.384	0.820	0.788	0.140	0.246	2.04
0.310	0.442	0.752	0.822	0.081	0.233	1.89
0.320	0.491	0.688	0.849	0.033	0.220	1.77
0.330	0.534	0.629	0.871	—0.010	0.208	1.67
0.340	0.572	0.574	0.889	—0.048	0.195	1.59
0.350	0.606	0.524	0.904	—0.082	0.183	1.53
0.360	0.638	0.477	0.917	—0.114	0.172	1.47
0.370	0.667	0.433	0.929	—0.143	0.160	1.42
0.380	0.695	0.392	0.938	—0.170	0.149	1.37
0.390	0.721	0.355	0.947	—0.196	0.138	1.33
0.400	0.745	0.319	0.955	—0.220	0.128	1.30
0.410	0.768	0.287	0.961	—0.242	0.118	1.26
0.420	0.791	0.257	0.967	—0.264	0.108	1.24
0.430	0.812	0.229	0.972	—0.284	0.098	1.21
0.440	0.832	0.203	0.977	—0.303	0.089	1.18
0.450	0.851	0.179	0.981	—0.322	0.080	1.16
0.460	0.870	0.157	0.984	—0.340	0.072	1.14
0.470	0.888	0.136	0.987	—0.357	0.064	1.12
0.480	0.906	0.117	0.990	—0.373	0.056	1.10
0.490	0.923	0.100	0.992	—0.389	0.049	1.09
0.500	0.940	0.084	0.994	—0.405	0.042	1.07
0.510	0.958	0.068	0.996	—0.420	0.035	1.06
0.520	0.972	0.055	0.997	—0.433	0.029	1.05
0.530	0.986	0.043	0.998	—0.446	0.023	1.04
0.540	1.001	0.031	0.999	—0.459	0.017	1.03
0.550	1.016	0.020	1.000	—0.472	0.011	1.02
0.560	1.031	0.010	1.000	—0.486	0.006	1.01
0.570	1.046	0.001	1.000	—0.499	0.001	1.00
0.571	1.047	0.000	1.000	—0.500	0.000	1.00

This solution is valid for $\sigma \leqslant 0$ or

$$\rho \geqslant \frac{\beta}{\sqrt{3}} \exp\left[\frac{1}{3}\left(1 - \frac{\pi}{2\sqrt{3}}\right)\right] = 0.361\beta$$

Consider now the annular region $\beta \leqslant \rho \leqslant 1$ where the thickness of the plate is constant. Here is valid the known solution satisfying $\sigma_r + \sigma_\theta = 0$ for $\rho = 1$

$$\left.\begin{matrix}\sigma_r \\ \sigma_\theta\end{matrix}\right\} = 2k\left(\ln\rho \mp \frac{1}{2}\right), \qquad w = v = 0, \qquad h = h_0$$

The value of $\rho = \beta$ corresponding to $\sigma_r = -2k$, $\sigma_\theta = 0$ is

$$\beta = \exp\left(-\frac{1}{2}\right) = 0.607$$

The results obtained by Hill are shown in Table 2.

TABLE 2.

ρ	$-\sigma_r / 2k$	$\sigma_\theta / 2k$	v	h / h_0
0.280	0.500	0.500	0.280	3.84
0.290	0.772	0.228	0.268	2.39
0.300	0.855	0.145	0.254	2.06
0.310	0.910	0.090	0.240	1.87
0.320	0.948	0.052	0.227	1.73
0.330	0.973	0.027	0.214	1.64
0.340	0.988	0.012	0.202	1.56
0.350	0.997	0.003	0.191	1.50
0.360	1.000	0.000	0.181	1.46
0.361	1.000	0.000	0.180	1.45

These results are valid only for that portion of the annular region $\alpha \leqslant \rho \leqslant \beta$ of varying thickness where $\sigma_\theta - \sigma_r = 2k$.

The values of σ_r and σ_θ as a function of ρ are shown by dotted lines in Fig. 3, and those of h by a dotted line in Fig. 4. σ_r is continuous in the whole interval $\alpha \leqslant \rho \leqslant 1$, and σ_θ is discontinuous at $\rho = \beta$. Comparison of solid and dotted curves in Figs. 3 and 4 indicates a considerable difference between the stress components σ_r and σ_θ, and also some difference in h.

In conclusion, we mention a solution of the same problem by Prager [2], where the Tresca plasticity condition and the associate flow law were used.

Equation (1) and the condition $\sigma_r = -2k$, $\sigma_\theta = 0$ in the plastic zone with variable thickness h, and (4) result in

$$\frac{dh}{d\rho}+\frac{h}{\rho}=0, \qquad \frac{v-\rho}{h}\frac{dh}{d\rho}+\frac{dv}{d\rho}+\frac{v}{\rho}=0$$

Since $v = 0$, $h = h_0$ for $\rho = \beta$, it is possible to find

$$v=\beta-\rho, \qquad h=h_0\frac{\beta}{\rho}$$

It is easy to see that

$$0\leqslant \varepsilon_\theta \leqslant -\varepsilon_r \quad \text{or} \quad 0\leqslant \beta/\rho-1\leqslant 1$$

so that

$$\beta/2\leqslant\rho\leqslant\beta$$

Comparison of these results with the previous solutions shows quite marked disagreement of the values of the stresses σ_r and σ_θ, as well as of the strain rates v and the thicknesses h.

BIBLIOGRAPHY

1. Hill, R., Plastic distortion of non-uniform sheets. *Phil. Mag.* Vol. 40, No. 309, 1949.

2. Prager, W., On the use of singular yield conditions and associated flow rules. *J. Appl. Mech.* Vol. 20, No. 3, 1953.

Translated by R.M. E.-I.